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A discrete dual finite volume method for the convection-diffusion equation: toward cold-plasma modeling.

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Abstract: To model cold plasma one needs to solve a system of convection-diffusion and Poisson equations as [3]. In [1], we proposed a discrete dual finite volume (DDFV) method to solve the Poisson equation with non-homogeneous jump conditions. This work is thus devoted to solving the convection-diffusion equation, in the framework of [2], using a self-consistent DDFV scheme to then apply it to cold-plasma modeling. Contrary to [2], mixed boundary conditions, and both the steady-state and time varying equations are studied. Numerical tests are proposed to show the convergence rate of the method in different configurations.

Keywords: convection-diffusion equation, finite-volume, DDFV, diamond scheme

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In this work, we focus on the modeling of cold plasma, which requires solving the following system of equations [3]:

$$\begin{aligned} \tfrac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) - D\nabla \cdot (\nabla n) - \alpha n|\mathbf{v}| &= 0 \\ \nabla \cdot (\nabla V) &= f, \end{aligned} \tag{1}$$

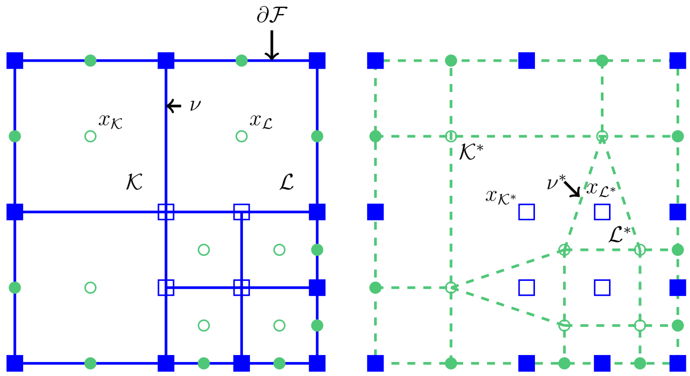
where $\mathbf{v} = \mu\mathbf{E}$, $\mathbf{E} = -\nabla V$, and D and α are given constants. Since the velocity field $\mathbf{v}$ is derived from the potential V, achieving high accuracy in solving the Poisson equation is crucial. In [1], a Discrete Duality Finite Volume (DDFV) scheme was proposed to solve this equation with high accuracy, even on distorted meshes.

Here, we extend this approach by focusing on the first equation. Specifically, we aim to develop a self-consistent DDFV scheme for the convection-diffusion equation:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) - D\nabla \cdot (\nabla n) - \gamma n = f, \tag{2}$$

where f is a source term, $\mathbf{v}$ is a given vector field, and γ is a constant. We also consider mixed boundary conditions. Our scheme relies on the DDFV direct discretization for the diffusion operator, as in the Poisson case, while the convection term is approximated using a finite volume approach based on the primal/dual mesh of DDFV. Figure 1 illustrates an example of this mesh structure, which is key to our discretization strategy. By leveraging the dual mesh and an affine interpolation of the field values on each edge, we construct a self-consistent approximation of the gradient.

We first validate our approach in steady-state configurations, demonstrating numerical convergence rates. In particular, we achieve second-order accuracy for the Dirichlet boundary case. Unlike [2], our formulation explicitly includes the term γn, which accounts for ionization effects in plasma modeling. We then extend our analysis to time-dependent cases,



(a) Primal mesh $\mathcal{M}$ (b) Dual mesh $\mathcal{M}^*$

Figure 1: Example of the primal and dual meshes when a quadrangular grid is considered. The circle corresponds to the primal unknowns while the square indicates the dual ones. The filling indicates boundary unknowns.

testing various time discretization schemes, including explicit and implicit Euler methods, as well as a θ-scheme.

This study highlights the potential of diamond-based schemes, such as DDFV, for solving multi-scale and multi-physics problems involving coupled equations. Our proposed self-consistent DDFV scheme for the convection-diffusion equation can be naturally combined with the Poisson solver from [1], providing a unified framework for cold plasma modeling.

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