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# A stochastic boundary element method for electromagnetic scattering in random media

Titouan Marquaille<sup>1,\*</sup>, Thomas Bonnafont<sup>1,2</sup>, Arnaud Coatanhay<sup>1</sup>

<sup>1</sup>Lab-STICC, UMR CNRS 6285, ENSTA, Institut Polytechnique de Paris, 29806 Brest, France

<sup>2</sup>DGA, 60 Bd. du général Martial Valin, Paris 75509, France

\*Email: titouan.marquaille@ensta.fr

## Abstract

The boundary element method (BEM) is standard for wave scattering, yet classical deterministic formulations neglect critical medium uncertainties. While Monte Carlo methods can quantify these effects, their slow statistical convergence leads to prohibitive costs. We propose a Stochastic BEM based on non-intrusive polynomial chaos expansion (PCE), which transforms the uncertainty quantification problem into independent deterministic BEM solves, enabling rapid computation of statistical moments with spectral convergence.

**Keywords :** Helmholtz equation, polynomial chaos expansion, proper orthogonal decomposition, electromagnetic scattering

## 1 Introduction

Electromagnetic scattering problems arise in antenna design or radar cross-section prediction where accurate field predictions are critical for system performance. The BEM is the standard numerical tool for exterior problems, as it discretizes only the scatterer's surface and naturally satisfies the Sommerfeld radiation condition at infinity.

However, deterministic models rely on idealized assumptions that neglect real-world variability. While geometric uncertainties (shape variations, surface roughness) have been extensively studied [3], material uncertainties, arising from manufacturing tolerances or environmental variations—pose distinct challenges, as they directly modify the governing wave equation. Variations in permittivity and permeability directly alter the effective wave number, shifting resonance frequencies and scattering signatures. Although Monte Carlo (MC) methods can quantify these effects, its slow convergence rate ( $\mathcal{O}(N^{-1/2})$ ) renders it unsuitable for high-fidelity solvers.

To overcome this bottleneck, we employ Polynomial Chaos Expansion (PCE). This method

approximates the random output as a series expansion in orthogonal stochastic polynomials, achieving spectral convergence for smooth parameter dependencies. Its non-intrusive formulation relies on regression and requires no modification of the deterministic BEM solver.

## 2 Problem Formulation

We consider the scattering of a time-harmonic ( $e^{j\omega t}$ ) transverse magnetic (TM) plane wave by a perfectly conducting (PEC) obstacle  $\Omega \subset \mathbb{R}^2$  embedded in a stochastic background medium. Let  $(\Theta, \mathcal{F}, \mathbb{P})$  be a probability space. The background relative permittivity  $\varepsilon_r(\theta)$  and permeability  $\mu_r(\theta)$  are modeled as independent random variables. Consequently, the total scalar electric field  $u(\mathbf{r}, \theta)$  satisfies :

$$\Delta u(\mathbf{r}, \theta) + k^2(\theta)u(\mathbf{r}, \theta) = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\Omega}, \quad (1)$$

where  $k(\theta) = k_0 \sqrt{\varepsilon_r(\theta)\mu_r(\theta)}$  is the random wave number. The field corresponds  $u = u^{\text{inc}} + u^{\text{sc}}$ , where  $u^{\text{inc}}(\mathbf{r}, \theta) = E_0 e^{jk(\theta)\mathbf{d} \cdot \mathbf{r}}$ , with  $\mathbf{d}$  the unit propagation vector. The scattered field  $u^{\text{sc}}$  satisfies the Sommerfeld radiation condition.

On the boundary  $\Gamma = \partial\Omega$ , the PEC condition  $u = 0$  leads to the electric field integral equation (EFIE). For a realization  $\theta$ , the induced surface current  $J$  satisfies:

$$\begin{aligned} & \frac{j\omega\mu_r(\theta)}{4} \int_{\Gamma} H_0^{(1)}(k(\theta)|\mathbf{r} - \mathbf{s}|) J(\mathbf{s}, \theta) d\Gamma(\mathbf{s}) \\ &= -u^{\text{inc}}(\mathbf{r}, \theta), \quad \forall \mathbf{r} \in \Gamma, \end{aligned} \quad (2)$$

where  $H_0^{(1)}(z)$  denotes the zeroth-order Hankel function of the first kind.

Applying the BEM with pulse basis functions and point matching yields the linear system  $\mathbf{Z}(\theta)\mathbf{J}(\theta) = \mathbf{V}(\theta)$ , where the impedance matrix  $\mathbf{Z}$  depends on the random parameter  $\theta$ . The quantity of interest is the complex surface current density  $\mathbf{J}(\theta) \in \mathbb{C}^N$ , essential for deriving the RCS through near-to-far-field transformation.

### 3 Stochastic Framework (PCE)

We place ourselves within the Hilbert space  $L^2(\Theta, \mathcal{F}, \mathbb{P})$  and define the vector of uncertain inputs as  $\mathbf{X}(\theta) = [\varepsilon_r(\theta), \mu_r(\theta)]^\top$  modeled as log-normal variables.

After an isoprobabilistic transformation of the log-normal inputs into a standard Gaussian space, a Hermite polynomial chaos expansion is used to represent the stochastic solution  $J \in L^2(\Theta)$ . We approximate  $J$  by a truncated expansion of order  $p$ :

$$J(\mathbf{r}, \theta) \approx \sum_{\alpha \in \mathcal{A}_p} J_\alpha(\mathbf{r}) \Psi_\alpha(\boldsymbol{\xi}(\theta)), \quad (3)$$

where  $\mathcal{A}_p = \{\alpha \in \mathbb{N}^2 : |\alpha| \leq p\}$ . The family  $\{\Psi_\alpha\}$  forms an orthogonal basis of multivariate polynomials [2]. To accommodate the complex-valued nature of the surface current and material parameters within the standard PCE framework, the solution  $J$  is decomposed into real and imaginary components, which are approximated as independent real-valued fields over the same polynomial basis.

Directly computing the expansion coefficients for each degree of freedom of the BEM mesh is computationally expensive. To remedy this, we employ proper orthogonal decomposition (POD) [1] to identify a reduced spatial basis  $\{\phi_k\}_{k=1}^K$  that optimally captures the energy of the system. The stochastic solution is then approximated as a linear combination of these few deterministic modes:

$$J(\mathbf{r}, \theta) \approx \bar{J}(\mathbf{r}) + \sum_{k=1}^K \beta_k(\theta) \phi_k(\mathbf{r}), \quad (4)$$

where  $\bar{J}$  is the mean field and  $\beta_k(\theta)$  are the stochastic reduced coordinates. Consequently, the PCE regression is performed solely on the  $K$  scalar coefficients  $\beta_k$  rather than the full discretized field, drastically reducing the computational burden while filtering spatial noise.

### 4 Numerical Results

The framework is validated on a PEC circular cylinder ( $R = 0.5$  m) illuminated at  $f = 300$  MHz. The background parameters have mean values  $\mu_\varepsilon = 2.5$  and  $\mu_\mu = 1$ .

To assess the robustness of the method, we test three distinct coefficients of variation  $CV = \frac{\text{std}}{\text{mean}} \in \{5\%, 20\%, 80\%\}$ . The Monte Carlo reference is computed using  $10^6$  realizations. A key

challenge is the nonlinear dependence of stochastic parameters in  $H_0^{(1)}(k(\theta)r)$ , which may affect convergence at high coefficient of variation. The error analysis for the complex surface current  $\mathbf{J}$  is summarized in Table 1.

| Moment                                   | $L^2$ Error (%)       | Max Abs. (%)          |
|------------------------------------------|-----------------------|-----------------------|
| <b>CV = 5%</b> (Speed-up $\times 1833$ ) |                       |                       |
| Mean                                     | $1.11 \times 10^{-6}$ | $2.64 \times 10^{-6}$ |
| Variance                                 | $2.87 \times 10^{-6}$ | $1.67 \times 10^{-5}$ |
| <b>CV = 20%</b> (Speed-up $\times 457$ ) |                       |                       |
| Mean                                     | $9.47 \times 10^{-5}$ | $2.32 \times 10^{-4}$ |
| Variance                                 | $4.58 \times 10^{-4}$ | $3.15 \times 10^{-3}$ |
| <b>CV = 80%</b> (Speed-up $\times 77$ )  |                       |                       |
| Mean                                     | $3.49 \times 10^{-3}$ | $9.42 \times 10^{-3}$ |
| Variance                                 | $8.66 \times 10^{-2}$ | $1.03 \times 10^{-1}$ |

Table 1: Relative  $L^2$  and maximum absolute errors (in %) of the PCE statistical moments compared to the Monte-Carlo reference.

The results show that the proposed PCE-POD framework remains accurate despite the strong kernel non-linearity. Although the regression error naturally scales with the input dispersion, precision remains excellent: even at  $CV = 80\%$ , the variance error is negligible ( $< 0.01\%$ ). This accuracy is obtained with speed-up factors ranging from  $\approx 1833$  to over 77 compared to the Monte Carlo reference. Consequently, the method offers a highly favorable cost-accuracy trade-off, validating its robustness for stochastic scattering problems involving large parameter variations.

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